Periodic Solutions of Marine Ecosystem Models of N-DOP type

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Outline

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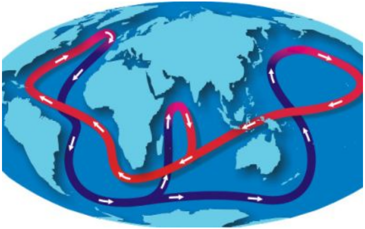
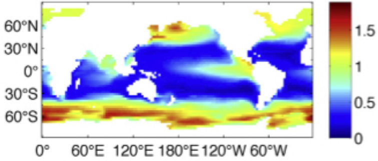
Motivation for ecosystem models

- Oceans are an important part of the global carbon cycle
- Carbon enters marine biogeochemical cycle
- Long-term storage of CO_2 on sea ground possible



Ocean circulation



Distribution of phosphate concentration

Model equations

Classical form of an ecosystem model equation of N-DOP type

$$
\begin{aligned}
\partial_t y_1 + \mathrm{div}(\mathbf{v}(t) y_1) - \mathrm{div}(\kappa \nabla y_1) - \lambda y_2 + d_1(y_1, y_2) &= 0 \quad \text{in } \Omega \times [0,T] \\
\partial_t y_2 + \underbrace{\mathrm{div}(\mathbf{v}(t) y_2)}_{\text{Advection}} - \underbrace{\mathrm{div}(\kappa \nabla y_2)}_{\text{Diffusion}} + \underbrace{\lambda y_2 + d_2(y_1, y_2)}_{\text{Reaction terms}} &= 0 \quad \text{in } \Omega \times [0,T] \\
\nabla y_j \cdot (\kappa \eta) + \overbrace{b_j(y_1, y_2)} &= 0 \quad \text{on } \Gamma \times [0,T],\ j = 1, 2
\end{aligned}
$$

with

- $\Omega \subset \mathbb{R}^3$ open, bounded, $\Gamma := \partial\Omega$
- $\mathbf{v} \in L^\infty(0,T; H^1(\Omega)^3)$ with $\mathrm{div}(\mathbf{v}(t)) = 0$ in Ω and $\mathbf{v}(t) \cdot \eta = 0$ on Γ
- $\kappa \in L^\infty(\Omega \times [0,T])$ with $\kappa_{\min} := \operatorname{ess\,inf}_{(x,t)} \kappa(x,t) > 0$
- d, b nonlinear reaction terms with $d(0) = 0 = b(0)$

Example: PO_4-DOP model by Parekh et al.

- Advection-diffusion-reaction equations:

$$\partial_t y_1 + \mathrm{div}(\mathbf{v}(t) y_1) - \mathrm{div}(\kappa \nabla y_1) - \lambda y_2 + d_1(y_1, y_2) = 0 \quad \text{in } \Omega \times [0, T]$$
$$\partial_t y_2 + \mathrm{div}(\mathbf{v}(t) y_2) - \mathrm{div}(\kappa \nabla y_2) + \lambda y_2 + d_2(y_1, y_2) = 0 \quad \text{in } \Omega \times [0, T]$$

- Nonlinear parts of reaction terms:

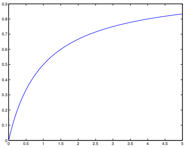
$$d_1(y) = \begin{cases} G(y_1), & \Omega_1 \text{ (upper layer)} \\ -(1-\nu)E(y_1), & \Omega_2 \text{ (lower layer)} \end{cases}$$

$$d_2(y) = \begin{cases} -\nu G(y_1), & \Omega_1, \\ 0, & \Omega_2 \end{cases}$$

- Typical nonlinear and nonlocal reaction terms

$G(y_1) = \alpha \frac{y_1}{|y_1| + K}$ (half saturation function)

$E(y) = \frac{\beta}{h_e} \left(\frac{x_3}{h_e} \right)^{-\beta - 1} \int_0^{h_e} G(y_1) dx_3$



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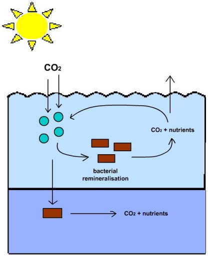
Conservation of mass



- Models represent closed cycles (no sources or sinks, cf. $d(0) = 0 = b(0)$)
- Mass of $y = (y_1, y_2)$ at point of time t

$$\text{mass}(y(t)) := \int_\Omega (y_1(t) + y_2(t))dx$$

- Solution $y = (y_1, y_2)$ has a constant mass iff $\frac{d}{dt}\,\text{mass}(y(t)) = 0$, equivalent to

$$\sum_{j=1}^{2}\left(\int_\Omega d_j(y,x,t)dx + \int_\Gamma b_j(y,s,t)ds\right) = 0 \quad \text{for almost all } t$$

$\rightarrow$ Determines boundary conditions

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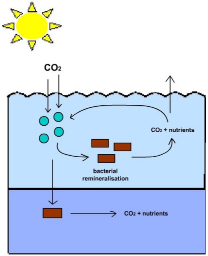

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Periodic solutions

- Pre-computed data κ and $\mathbf{v}$ are annually periodic ("climatological data"):

$$\kappa(t) = \kappa(t+T) \quad \text{and} \quad \mathbf{v}(t) = \mathbf{v}(t+T), \quad t \geq 0$$

- Periodic concentration $y := (y_1, y_2)$ of modeled substances desired, i.e.

$$y_j(0) = y_j(T), \qquad j = 1, 2$$

- Computation with fixed point iteration (pseudo-time stepping or "spin-up")
  - Find fixed point of the map Φ corresponding to one year model time
  - Method: Repeat $y_{i+1} = \Phi(y_i)$ long enough
    (y_i=vector of all concentrations at the end of year $i = 0, 1, \dots$)
  - Theoretical confirmation for this procedure?

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Weak formulation (1)

Multiply classical model equation by test function in order to reduce the requirements for a solution

Resulting weak formulation, e.g. for the first equation (w_1 test function)

$$\int_0^T \{\langle y_1', w_1\rangle + \int_\Omega \{\kappa \nabla y_1 \cdot \nabla w_1 + \mathrm{div}(\mathbf{v} y_1) w_1 - \lambda y_2 w_1 + d_1(y) w_1\} dx + \int_\Gamma b_1(y) w_1 ds\} dt = 0$$

Space of weak solutions:

$$W := W(0,T;H^1(\Omega)) := \{v \in L^2(0,T;H^1(\Omega)); v' \in L^2(0,T;H^1(\Omega)^*)\}$$

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Weak formulation (2)

Alternative view on the summands

$$B, F_1, F_2 : L^2(0,T;H^1(\Omega)) \to L^2(0,T;H^1(\Omega)^*), \text{ defined by}$$

$$\langle B(y), v\rangle := \int_0^T \int_\Omega \{\kappa \nabla y \cdot \nabla v + \operatorname{div}(\mathbf{v}y)v\} dx dt,$$
$$\langle F_j(y), v\rangle := \int_0^T \{-\int_\Omega d_j(y) v dx - \int_\Gamma b_j(y) v ds\} dt$$

for $j = 1, 2$ and $y, v \in L^2(0,T;H^1(\Omega))$

Problem in the dual space

$$y_1' + B(y_1) - \lambda y_2 = F_1(y)$$
$$y_2' + B(y_2) + \lambda y_2 = F_2(y)$$
$$y(0) = y(T)$$

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y_2' + B(y_2) + \lambda y_2 &= F_2(y) \\
y(0) &= y(T)
\end{aligned}$$

Typical result for periodic solvability

Existence theorem of Gajewski et al., 1974

Let $V \subset H \subset V^*$ be an evolution triple , $X := L^2(0,T;V)$. If $A : X \to X^*$ is a continuous, monotone and coercive operator, the problem

$$u' + Au = f, \quad u(0) = u(T),$$

has a solution $u \in W(0,T;V) \hookrightarrow C([0,T];H)$ for every $f \in X^*$.

Definitions:

- A monotone $:\Leftrightarrow \langle Au - Av, u - v\rangle \geq 0$ for all $u, v \in X$
- A coercive $:\Leftrightarrow \frac{\langle Au,u\rangle}{\|u\|_X} \to \infty$ if $\|u\|_X \to \infty$

Problem: Coercivity usually not fulfilled by ecosystem model equations

Outline

Existence theorem

Theorem

Let $\lambda > 0$ *and* $C \in \mathbb{R}$. *Under the assumptions*

- d, b *are continuous*
- $\sum_{j=1}^{2} \left(\int_{\Omega} d_j(y,x,t)dx + \int_{\Gamma} b_j(y,s,t)ds \right) = 0$ *for almost all* t *and all* y
- $\max\{|d_j(y,x,t)|, |b_j(y,x,t)|\} \leq M$ *for all* y, $j = 1, 2$ *and almost all* (x,t)

there is at least one periodic weak solution $y \in W^2$ *of the ecosystem model equation with*

$$\mathrm{mass}(y(t)) = C \quad \text{for all } t \in [0, T].$$

In particular, there are nontrivial solutions.

Proof: Strategy

1. For every $z \in L^2(\Omega \times [0,T])^2$, solve

$$\begin{aligned} y_1' + B(y_1) - \lambda y_2 &= F_1(z) \\ y_2' + B(y_2) + \lambda y_2 &= F_2(z) \\ y(0) &= y(T) \\ \mathrm{mass}(y(t)) &= C. \end{aligned}$$

2. Find a fixed point of the map $z \mapsto y$ where y is the solution of Step 1 (Schauder's fixed point theorem).

Open questions

- Proved: Marine ecosystem models of N-DOP type have nontrivial periodic solutions

- No statement about fixed point iteration
  1. Convergence?
  2. Solution uniquely determined by initial mass?

- Future work: Investigation of actual fixed point iteration (problem discretized w.r.t. time)