Periodic Solutions of Marine Ecosystem Models of $N$-DOP type

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Outline

1 Introduction

2 Characteristic properties

3 The existence theorem
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1 Introduction

2 Characteristic properties

3 The existence theorem
Motivation for ecosystem models

- Oceans are an important part of the global carbon cycle
- Carbon enters marine biogeochemical cycle
- Long-term storage of $CO_2$ on sea ground possible

Ocean circulation

Distribution of phosphate concentration
Model equations

Classical form of an ecosystem model equation of \( N-DOP \) type

\[
\begin{align*}
\partial_t y_1 + \text{div}(\mathbf{v}(t)y_1) - \text{div}(\kappa \nabla y_1) - \lambda y_2 + d_1(y_1, y_2) &= 0 & \text{in } \Omega \times [0, T] \\
\partial_t y_2 + \text{div}(\mathbf{v}(t)y_2) - \text{div}(\kappa \nabla y_2) + \lambda y_2 + d_2(y_1, y_2) &= 0 & \text{in } \Omega \times [0, T]
\end{align*}
\]

Advection \hspace{1cm} Diffusion \hspace{1cm} Reaction terms

\[
\nabla y_j \cdot (\kappa \eta) + b_j(y_1, y_2) = 0 \quad \text{on } \Gamma \times [0, T], \ j = 1, 2
\]

with

- \( \Omega \subset \mathbb{R}^3 \) open, bounded, \( \Gamma := \partial \Omega \)
- \( \mathbf{v} \in L^\infty(0, T; H^1(\Omega)^3) \) with \( \text{div}(\mathbf{v}(t)) = 0 \) in \( \Omega \) and \( \mathbf{v}(t) \cdot \eta = 0 \) on \( \Gamma \)
- \( \kappa \in L^\infty(\Omega \times [0, T]) \) with \( \kappa_{\text{min}} := \text{ess inf}_{(x,t)} \kappa(x, t) > 0 \)
- \( d, b \) nonlinear reaction terms with \( d(0) = 0 = b(0) \)
Example: $PO_4-DOP$ model by Parekh et al.

- Advection-diffusion-reaction equations:

  \[
  \begin{align*}
  \partial_t y_1 + \text{div}(\mathbf{v}(t)y_1) - \text{div}(\kappa \nabla y_1) - \lambda y_2 + d_1(y_1, y_2) &= 0 \quad \text{in } \Omega \times [0, T] \\
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  \end{align*}
  \]

- Nonlinear parts of reaction terms:

  \[
  d_1(y) = \begin{cases} 
  G(y_1), & \Omega_1 \ (\text{upper layer}) \\
  -(1 - \nu)E(y_1), & \Omega_2 \ (\text{lower layer}) 
  \end{cases}
  \]

  \[
  d_2(y) = \begin{cases} 
  -\nu G(y_1), & \Omega_1, \\
  0, & \Omega_2
  \end{cases}
  \]

- Typical nonlinear and nonlocal reaction terms

  \[
  G(y_1) = \alpha \frac{y_1}{|y_1|+K} \quad \text{(half saturation function)}
  \]

  \[
  E(y) = \frac{\beta}{h_e} \left( \frac{x_3}{h_e} \right)^{-\beta-1} \int_0^{h_e} G(y_1) \, dx_3
  \]
1. Introduction

2. Characteristic properties

3. The existence theorem
Conservation of mass

- Models represent closed cycles (no sources or sinks, cf. \( d(0) = 0 = b(0) \))

- Mass of \( y = (y_1, y_2) \) at point of time \( t \)

\[
\text{mass}(y(t)) := \int_{\Omega} (y_1(t) + y_2(t)) \, dx
\]

- Solution \( y = (y_1, y_2) \) has a constant mass iff \( \frac{d}{dt} \text{mass}(y(t)) = 0 \), equivalent to

\[
\sum_{j=1}^{2} \left( \int_{\Omega} d_j(y, x, t) \, dx + \int_{\Gamma} b_j(y, s, t) \, ds \right) = 0 \quad \text{for almost all } t
\]

→ Determines boundary conditions
Conservation of mass

- Models represent closed cycles (no sources or sinks, cf. $d(0) = 0 = b(0)$)

- Mass of $y = (y_1, y_2)$ at point of time $t$

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  → Determines boundary conditions
Pre-computed data $\kappa$ and $\nu$ are annually periodic ("climatological data"): 

$$\kappa(t) = \kappa(t + T) \quad \text{and} \quad \nu(t) = \nu(t + T), \quad t \geq 0$$

Periodic concentration $y := (y_1, y_2)$ of modeled substances desired, i.e.

$$y_j(0) = y_j(T), \quad j = 1, 2$$

Computation with fixed point iteration (pseudo-time stepping or "spin-up")

- Find fixed point of the map $\Phi$ corresponding to one year model time

- Method: Repeat $y_{i+1} = \Phi(y_i)$ long enough
  ($y_i =$ vector of all concentrations at the end of year $i = 0, 1, \ldots$)

Theoretical confirmation for this procedure?
Periodic solutions

- Pre-computed data $\kappa$ and $v$ are annually periodic ("climatological data"): 
  \[ \kappa(t) = \kappa(t + T) \quad \text{and} \quad v(t) = v(t + T), \quad t \geq 0 \]

- Periodic concentration $y := (y_1, y_2)$ of modeled substances desired, i.e.
  \[ y_j(0) = y_j(T), \quad j = 1, 2 \]

- Computation with fixed point iteration (pseudo-time stepping or "spin-up")
  - Find fixed point of the map $\Phi$ corresponding to one year model time
  - Method: Repeat $y_{i+1} = \Phi(y_i)$ long enough 
    ($y_i$=vector of all concentrations at the end of year $i = 0, 1, \ldots$)

- Theoretical confirmation for this procedure?
Weak formulation (1)

Multiply classical model equation by test function in order to reduce the requirements for a solution

Resulting weak formulation, e.g. for the first equation ($w_1$ test function)

$$\int_0^T \{ \langle y'_1, w_1 \rangle + \int \{ \kappa \nabla y_1 \cdot \nabla w_1 + \text{div}(v y_1) w_1 - \lambda y_2 w_1 + d_1(y) w_1 \} dx \\ + \int_{\Gamma} b_1(y) w_1 ds \} dt = 0$$

Space of weak solutions:

$$W := W(0,T; H^1(\Omega)) := \{ v \in L^2(0,T; H^1(\Omega)); v' \in L^2(0,T; H^1(\Omega)^*) \}$$
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\[ + \int_{\Gamma} b_1(y)w_1 ds \} dt = 0 \]

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Weak formulation (2)

Alternative view on the summands

\[ B, F_1, F_2 : L^2(0, T; H^1(\Omega)) \to L^2(0, T; H^1(\Omega)^*) , \text{ defined by} \]

\[
\langle B(y), v \rangle := \int_0^T \int_\Omega \{ \kappa \nabla y \cdot \nabla v + \text{div}(v y) v \} \, dx \, dt,
\]

\[
\langle F_j(y), v \rangle := \int_0^T \{ - \int_\Omega d_j(y) v \, dx - \int_\Gamma b_j(y) v \, ds \} \, dt
\]

for \( j = 1, 2 \) and \( y, v \in L^2(0, T; H^1(\Omega)) \)

Problem in the dual space

\[
y_1' + B(y_1) - \lambda y_2 = F_1(y)
\]

\[
y_2' + B(y_2) + \lambda y_2 = F_2(y)
\]

\[
y(0) = y(T)
\]
Weak formulation (2)

Alternative view on the summands

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for \( j = 1, 2 \) and \( y, v \in L^2(0, T; H^1(\Omega)) \)

Problem in the dual space

\[ \begin{align*}
y_1' + B(y_1) - \lambda y_2 &= F_1(y) \\
y_2' + B(y_2) + \lambda y_2 &= F_2(y) \\
y(0) &= y(T)
\end{align*} \]
Existence theorem of Gajewski et al., 1974

Let $V \subset H \subset V^*$ be an evolution triple, $X := L^2(0, T; V)$. If $A : X \to X^*$ is a continuous, monotone and coercive operator, the problem

$$u' + Au = f, \quad u(0) = u(T),$$

has a solution $u \in W(0, T; V) \hookrightarrow C([0, T]; H)$ for every $f \in X^*$.

Definitions:

- $A$ monotone $\iff \langle Au - Av, u - v \rangle \geq 0$ for all $u, v \in X$

- $A$ coercive $\iff \frac{\langle Au, u \rangle}{\lVert u \rVert_X} \to \infty$ if $\lVert u \rVert_X \to \infty$

Problem: Coercivity usually not fulfilled by ecosystem model equations
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Existence theorem

Theorem

Let $\lambda > 0$ and $C \in \mathbb{R}$. Under the assumptions

- $d, b$ are continuous
- $\sum_{j=1}^{2} (\int_{\Omega} d_j(y, x, t) dx + \int_{\Gamma} b_j(y, s, t) ds) = 0$ for almost all $t$ and all $y$
- $\max\{|d_j(y, x, t)|, |b_j(y, x, t)|\} \leq M$ for all $y, j = 1, 2$ and almost all $(x, t)$

there is at least one periodic weak solution $y \in W^2$ of the ecosystem model equation with

$$\text{mass}(y(t)) = C \quad \text{for all } t \in [0, T].$$

In particular, there are nontrivial solutions.
1. For every \( z \in L^2(\Omega \times [0, T])^2 \), solve

\[
\begin{align*}
y_1' + B(y_1) - \lambda y_2 &= F_1(z) \\
y_2' + B(y_2) + \lambda y_2 &= F_2(z) \\
y(0) &= y(T) \\
\text{mass}(y(t)) &= C.
\end{align*}
\]

2. Find a fixed point of the map \( z \mapsto y \) where \( y \) is the solution of Step 1 (Schauder’s fixed point theorem).
Open questions

- Proved: Marine ecosystem models of \( N-DOP \) type have nontrivial periodic solutions

- No statement about fixed point iteration
  1. Convergence?
  2. Solution uniquely determined by initial mass?

- Future work: Investigation of actual fixed point iteration (problem discretized w.r.t. time)